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Some properties of the perturbed Haar wavelets

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Abstract

One of the authors has studied the properties of a family of Riesz bases obtained by
perturbing the Haar function using B–splines. Although these bases cannot be obtained
by multiresolution analyses, they have other interesting properties. The present paper
discusses how a discrete signal $\{a_r; 0 \leq r \leq N - 1\}$ can be studied by considering
a suitable function of the form $f(t) := \sum_{r=0}^{N-1} a_r f_r(t)$, so that the existing theory for
functions defined over a continuous domain can be applied.

1   Introduction

In what follows $\mathbf{Z}$ will denote the integers and $\mathbb{R}$ the real numbers; t and x will always
denote real variables. The support of a function f will be denoted by $\mathrm{supp}\,(f)$, its
quadratic norm by $\|f\|$ and if $f \in L(\mathbb{R})$ its Fourier transform is defined by

$$\widehat{f}(x) := \int_{\mathbb{R}} e^{-txi} f(t)\, dt.$$

In [3] we found a family of affine wavelet Riesz bases of $L^2(\mathbb{R})$, of bounded support and
arbitrary degrees of smoothness, obtained by smoothing the discontinuities of the Haar
function using B–splines. Although these bases are not orthogonal they are symmetric,
a feature that is lacking in orthogonal wavelets. Our bases can be constructed so that
the difference between the frame bounds (which are given explicitly) can be made as
small as desired. In general, orthogonal wavelets are represented by infinite series, and
for computational purposes values are generated over a discrete set using the cascade
algorithm [2, 5]. Our bases, on the other hand, are given in closed form. We now briefly
describe how these wavelets are defined and introduce additional notation and make
assumptions that will be used in the subsequent discussion.

Let $N_m(t)$ denote the B–spline of order m $(m \geq 2)$ ([1], Chapter 4), $\chi_{[0,m-1]}(t)$ the

characteristic function of $[0, m-1]$),

$$g(t) := \chi_{[0,m-1]}(t) \sum_{k=0}^{m-2} N_m(t-k), \quad g_1(t) := g(t-m+1), \quad h(t) := (1/2) \sum_{k=0}^{m-2} N_m(t-k),$$

and $q(t) := g_1(t) - h(t)$. For $0 < \delta < 1/2$, let $\alpha_1 = -\alpha_2 = -\alpha_3 = \alpha_4 = 2(m-1)/\delta$, $\beta_1 = 2(m-1)$, $\beta_2 = 2(m-1)(1+\delta)/\delta$, $\beta_3 = -\beta_4 = (m-1)/\delta$,

$$p^{\{i\}}(t) := (-1)^{i-1} q(\alpha_i t + \beta_i), i = 1, 2, 3, 4, \quad p^{\{5\}}(t) := -(\chi_{[1/2-\delta,1/2)}(t) - \chi_{[1/2,1/2+\delta)}(t)),$$

$$p^{\{6\}}(t) := \chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t), \quad \text{and} \quad \psi(t) := \sum_{i=1}^{6} p^{\{i\}}(t).$$

We will call ψ *the perturbed Haar wavelet*. In [3] we proved that supp $(\psi) \subseteq [-\delta, 1 + \delta]$, $\psi \in C^{m-2}(\mathbb{R})$, and that if $\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k)$, then $\{\psi_{j,k}; j, k \in \mathbf{Z}\}$ is a Riesz basis, and we provided explicit upper and lower frame bounds. Moreover, in [7] we showed that given a function μ, the wavelet coefficients $\langle \mu, \psi_{j,k} \rangle$ can be computed in $O(N)$ steps (where N is the sample size), just as in the orthogonal case.

In this paper we will discuss the application of the perturbed Haar wavelet to the study of discrete signals. Let us first look at the orthogonal case for comparison.

Let μ be an orthogonal wavelet associated with a multiresolution analysis $\{V_j; j \in \mathbf{Z}\}$ and a scaling function ϕ, with the caveat that the definition of multiresolution analysis that we are adopting is that of [1] and [4], and therefore $V_j \subset V_{j+1}, j \in \mathbf{Z}$, whether other authors, like [2] and [5] assume that $V_{j+1} \subset V_j$. If $\mathbf{a} := \{a_r; 0 \le r \le N-1\}$ is an arbitrary sequence of real or complex numbers, then this discrete signal is transformed into a continuous one by considering the function $\nu(t) := \sum_{r=0}^{N-1} a_r \phi(t-r)$.

The study of the signal $\nu(t)$ has two stages: the *analysis* stage consists in computing the wavelet coefficients, whereas the *synthesis* stage consists in reconstructing the signal from the wavelet coefficients. If W_j denotes the closure of the linear span of the functions $\mu_{j,k}$, $j \in \mathbf{Z}$, then the W_j are mutually orthogonal and $V_0 = \oplus_{j<0} W_j$. Since $\nu \in V_0$, it turns out that the wavelet coefficients $\langle \nu, \mu_{j,k} \rangle$ vanish for $j > 0$. Moreover, since $\nu(t)$ has compact support, for each $j \le 0$ there is only a finite number of nonzero wavelet coefficients.

With the perturbed Haar wavelet we face an additional problem: the spaces W_j are no longer orthogonal, and we can therefore no longer assume that all the wavelet coefficients corresponding to positive values of j must vanish. Moreover, we may not even have a scaling function: in [8] we showed that if $\delta = 2^\ell$, where ℓ is a negative integer, then the perturbed Haar wavelet ψ that corresponds to this value of δ cannot be generated by a multiresolution analysis.

To overcome these difficulties, we proceed as follows. Let $n \in \mathbf{Z}$ be such that $2^n \ge 4(m-1)$, $b^{\{1\}}(t) := \chi_{[0,2(m-1))}(t)q(t)$, $b^{\{2\}}(t) := q(4(m-1)-t)$, $b(t) := b^{\{1\}}(t) + b^{\{2\}}(t)$, $f_r(t) := a_r b(2^n t - 4(m-1)r)$, and $f(t) := \sum_{r=0}^{N-1} f_r(t)$. By a direct application of [3] Lemma 6 we obtain the following

Lemma 1 *The function $b(t)$ has the following properties:*
  (a) *supp*$(b) \subseteq [0, 4(m-1)]$,   (b) $b \in C^{m-2}(\mathbb{R})$,   (c) $b(2(m-1)) = 1$,

(d) $\frac{d^k}{dx^k}b(0) = \frac{d^k}{dx^k}b(2(m-1)) = \frac{d^k}{dx^k}b(4(m-1)) = 0,\ 1 \le k \le m-2,$
(e) *The total variation of b does not exceed* $4(m-1)$,   (f) $|b(t)| \le 1$.

From the preceding lemma we conclude that $\operatorname{supp}(f) \subseteq [0,1]$, and that the functions f_r have disjoint supports. This implies that $\|f\|^2 = \|b\|^2\|\mathbf{a}\|^2 2^{-n}$, where $\|\mathbf{a}\|^2 := \sum_{r=0}^{N-1}|a_r|^2$. We will also use the ℓ_1 norm: $\|\mathbf{a}\|_1 := \sum_{r=0}^{N-1}|a_r|$. Note, moreover, that $f \in C^{m-2}(\mathbb{R})$, and that $f(2^{1-n}(m-1)(2r+1)) = f_r(2^{1-n}(m-1)(2r+1)) = a_r b(2(m-1)) = a_r$.

In theory, given all its wavelet coefficients, the function f can be reconstructed using the frame algorithm or other, even faster, algorithms [5]. However, since there may be an infinite number of nonzero wavelet coefficients, the application of such algorithms may not always be practical. We will adopt an approximation approach. If $A = A(\delta, m)$, and $B = B(\delta, m)$ are respectively the lower and upper frame bounds of the Riesz basis generated by ψ, $h_{j,k} := \langle f, \psi_{j,k}\rangle$, and $Lf := \sum_{j,k \in \mathbf{Z}} h_{j,k}\psi_{j,k}$, then from the error estimates for the frame algorithm we know that $\|Lf - f\| \le ((B - A)/(B + A))\|f\|$. Since, as remarked above, we can make A and B as close to 1 as we want by making δ sufficiently small, we conclude that for every $\varepsilon > 0$ there is a δ_0 such that if $0 < \delta < \delta_0$, then $\|Lf - f\| < \varepsilon\|f\|$. To approximate f using the wavelet coefficients it will therefore suffice to approximate Lf by an operator of the form

$$Ef = \sum_{j=j_1}^{j_2} \sum_{k \in \mathbf{Z}} h_{j,k}\psi_{j,k}.$$

Observe that since f has bounded support, Ef reduces to a finite sum.

Our objective will be accomplished by showing that there is a constant K such that

$$\left\|\sum_{k \in \mathbf{Z}} h_{j,k}\psi_{j,k}\right\| \le K\|\mathbf{a}\|\, 2^{-|j|/2}.$$

But first we need to prove five lemmas, of some independent interest. We begin with

Lemma 2 *Let* $\{a_h; k \in \mathbf{Z}\}$ *and* $\{b_h; k \in \mathbf{Z}\}$ *be increasing sequences such that* $a_k < b_{k-1} < a_{k+1},\ k \in \mathbf{Z}$. *Assume that* $f_k \in L^2(\mathbb{R})$ *and that* $\operatorname{supp}(f_k) \subseteq [a_k, b_k]$, *and let* $f := \sum_{k \in \mathbf{Z}} f_k$. *Then* $\|f\|^2 \le 2\sum_{k \in \mathbf{Z}}\|f_k\|^2$.

Proof: If $r < k-1$ then $b_r \le b_{k-2} < a_k$, whereas if $r > k+1$ then $a_r \ge a_{k+2} > b_k$. This implies that if $r \ne k-1, k$ then $f_r(t) = 0$ on $[a_k, b_k]$, and we readily see that

$$\|f\|^2 \le 2\sum_{k \in \mathbf{Z}}\int_{a_k}^{b_k}|f_k(t)|^2 = 2\sum_{k \in \mathbf{Z}}\|f_k(t)\|^2. \qquad \square$$

Lemma 3 *Let* $u \in L^2(\mathbb{R})$ *be a function with support in an interval* $[a, b]$ *with* $b - a \le 1$. *If* $j \le 0$, *then*

$$\sum_{k \in \mathbf{Z}}|\langle u, \psi_{j,k}\rangle|^2 \le 3\|u\|^2 2^j.$$

Proof: Let $j \le 0$ be arbitrary but fixed, and define $I(k) := \operatorname{supp}(\psi_{j,k}) \cap [a, b]$. Then $I(k) \subseteq [2^{-j}(k - \delta), 2^{-j}(k + \delta + 1)] \cap [a, b]$. If $I(k) = \emptyset$ then, either $2^{-j}(k + \delta + 1) \le a$,

or $2^{-j}(k - \delta) \geq b$. This implies that if $I(k) \neq \emptyset$, then $k \in (2^j a - \delta - 1, 2^j b + \delta)$. Since the length of this interval is less than 3, we conclude that there are at most three values of k for which $I(k) \neq \emptyset$. In other words, there are at most three values of k for which $h_{j,k} \neq 0$. Since $|\psi(t)| \leq 1$, for any such k we have:

$$|\langle u, \psi_{j,k}\rangle|^2 \;=\; 2^j \left|\int_{I(k)} u(t)\psi(2^k t - k)\right|^2 dt \leq 2^j \int_{I(k)} |u(t)|^2\, dt \int_{I(k)} |\psi(2^k t - k)|^2\, dt$$

$$\leq \;(b-a)2^j \int_{I(k)} |u(t)|^2\, dt \leq 2^j\|u\|^2. \qquad \square$$

Lemma 4 *Let $\alpha, \beta, \gamma, \sigma \in I\!\!R$, with $\alpha, \gamma \neq 0$, and define $c(t) := q(\alpha t + \beta)$, $d(t) := q(\gamma t + \sigma)$, and*

$$K = 2\left\{\left[25/64 + (25/192)^{2/3}\right](m-1)^4 + (m-1)^2/1024\right\}.$$

If $j > 0$ and $i = 5, 6$, then

(a) $\displaystyle\sum_{k\in\mathbf{Z}} |\langle d, c_{j,k}\rangle|^2 \leq 2\left(4\sqrt{K}\,\alpha^{-2} + 1/3\right)^2 2^{-j}$; (b) $\displaystyle\sum_{k\in\mathbf{Z}} \left|\langle d, p_{j,k}^{(i)}\rangle\right|^2 \leq \left(2\sqrt{2} + 1/2\right)^2 2^{-j}$.

Proof: (a) From [3] p. 3367 (bearing in mind the slightly different definition of the Fourier transform), we have

$$\widehat{g}(x) = (i/x)e^{-(m-1)xi/2}\left[e^{-(m-1)xi/2} - ((2/x)\sin x/2)^{m-1}\right].$$

From [1] p. 56 (3.2.16),

$$\widehat{N}_m(x) = e^{-(1/2)mxi}[(2/x)\sin x/2]^m. \qquad (1.1)$$

Let

$$s(x) := [(2/x)\sin x/2]^{m-1}.$$

Then

$$\widehat{g}_1(x) = e^{-(m-1)xi}\,\widehat{g}(x) = (i/x)[e^{-2(m-1)xi} - e^{-(3/2)(m-1)xi}s(x)].$$

Since

$$\widehat{h}(x) = \frac{1}{2}\sum_{k=0}^{m-2} e^{-kxi}\widehat{N}_m(x) = (1/2)\frac{1 - e^{-(m-1)xi}}{1 - e^{-xi}}\widehat{N}_m(x),$$

a straightforward computation yields

$$\widehat{h}(x) = -i/(2x)[e^{-(1/2)(m-1)xi} - e^{-(3/2)(m-1)xi}]s(x),$$

whence

$$\widehat{q}(x) \;=\; i\frac{1}{x}e^{-(m-1)xi}[\cos(m-1)x - s(x)\cos\frac{1}{2}(m-1)x$$

$$+i(2s(x)\sin\frac{1}{2}(m-1)x - \sin(m-1)x)]. \qquad (1.2)$$

This implies that

$$|\widehat{q}(x)|^2 \leq 8x^{-2}, \quad x \neq 0. \qquad (1.3)$$

On the other hand,

$$\widehat{q}(x) = ix^{-1}e^{-(m-1)xi}\left[(v_1 + v_2) + i(v_3 + v_4)\right],$$

where

$$v_1 := \cos(m-1)x - \cos(1/2)(m-1)x, \qquad v_2 := [1 - s(x)]\cos(1/2)(m-1)x,$$

$$v_3 := s(x)\left[2\sin(1/2)(m-1)x - \sin(m-1)x\right], \qquad v_4 := [s(x) - 1]\sin(m-1)x.$$

A McLaurin expansion shows that $|v_1| \leq (5/8)(m-1)^2 x^2$. Since $1 - u^{m-1} = (1 - u)\sum_{k=0}^{m-2} u^k$ and $|\sin u| \leq |u|$, we infer that

$$|1 - s(x)| \leq (m-1)|1 - (2/x)\sin x/2| = (m-1)(2/x)|x/2 - \sin x/2|.$$

Since $|u - \sin u| \leq |u|^3/6$, we conclude that $|1 - s(x)| \leq (m-1)x^2/48$. Thus,

$$|v_2(x)| \leq (m-1)x^2/48, \quad \text{and} \quad |v_4(x)| \leq (m-1)x^2/48.$$

Another McLaurin expansion yields $|v_3| \leq (5/24)(m-1)^3|x|^3$. Clearly $|v_3| \leq 3$; thus $|v_3| = |v_3|^{2/3}|v_3|^{1/3} \leq (25/192)^{1/3}(m-1)^2 x^2$. Since

$$|\widehat{q}(x)|^2 = x^{-2}[(v_1 + v_2)^2 + (v_3 + v_4)^2] \leq 2x^{-2}[v_1^2 + v_2^2 + v_3^2 + v_4^2],$$

we deduce that

$$|\widehat{q}(x)|^2 \leq Kx^2. \tag{1.4}$$

From Plancherel's identity we have:

$$
\begin{aligned}
\langle d, c_{j,k}\rangle &= 2^{j/2}\int_{\mathbb{R}} d(t)c(2^j t - k)\,dt = 2^{j/2}/(2\pi)\int_{\mathbb{R}} e^{kxi}\widehat{c}(x)\widehat{d}(2^j x)\,dx \\
&= 2^{j/2}/(2\pi)\int_0^{2\pi} e^{kxi}\sum_{k\in\mathbb{Z}}\widehat{c}(x + 2\pi r)\widehat{d}(2^j(x + 2\pi r))\,dx.
\end{aligned}
$$

This means that $\{2^{-j/2}\langle d, c_{j,k}\rangle; k \in \mathbb{Z}\}$ is the sequence of Fourier coefficients of the function $\sum_{k\in\mathbb{Z}}\widehat{c}(x + 2\pi r)\widehat{d}(2^j(x + 2\pi r))$. Thus, applying Bessel's identity and then the Cauchy–Schwarz inequality twice (once for sums and once for integrals), we have:

$$2\pi 2^{-j}\sum_{k\in\mathbb{Z}}|\langle d, c_{j,k}\rangle|^2 = \int_0^{2\pi}\left|\sum_{r\in\mathbb{Z}}\widehat{c}(x + 2\pi r)\widehat{d}(2^j(x + 2\pi r))\right|^2 dx$$

$$\leq \int_0^{2\pi}\left[|\widehat{c}(x)\widehat{d}(2^j x)| + |\widehat{c}(x - 2\pi)\widehat{d}(2^j(x - 2\pi))| + \left|\sum_{r\neq 0,-1}\widehat{c}(x + 2\pi r)\widehat{d}(2^j(x + 2\pi r))\right|\right]^2 dx$$

$$\leq \left[\left(\int_0^{2\pi}|\widehat{c}(x)\widehat{d}(2^j x)|^2\,dx\right)^{1/2} + \left(\int_0^{2\pi}|\widehat{c}(x - 2\pi)\widehat{d}(2^j(x - 2\pi))|^2\,dx\right)^{1/2}\right.$$

$$\left.+ \left(\int_0^{2\pi}|\sum_{r\neq 0,-1}\widehat{c}(x + 2\pi r)\widehat{d}(2^j(x + 2\pi r))|^2\,dx\right)^{1/2}\right]^2$$

$$=: \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2.$$

Since $\widehat{c}(x) = \alpha^{-1}e^{(\beta/\alpha)xi}\widehat{q}(\alpha^{-1}x)$, (1.3) implies that

$$|\widehat{c}(x + 2\pi r)|^2 \le 8|x + 2\pi r|^{-2}, \quad x \ne 2\pi r, \tag{1.5}$$

whereas from (1.4) we see that

$$|\widehat{c}(x + 2\pi r)|^2 \le K\,\alpha^{-4}|x + 2\pi r|^2. \tag{1.6}$$

Since $\widehat{d}(x) = \gamma^{-1}e^{(\tau/\gamma)xi}\widehat{q}(\gamma^{-1}x)$, (1.3) also implies that

$$|\widehat{d}(2^j(x + 2\pi r))|^2 \le 4^{-j+1}\,2|x + 2\pi r|^{-2}, \quad x \ne 2\pi r. \tag{1.7}$$

Since S_1 is obtained by integrating the product of the left–side members of (1.6) and (1.7) (with $r = 0$) over an interval of length 2π, we readily see that

$$S_1 \le 16\pi K\,\alpha^{-4}4^{-j}. \tag{1.8}$$

A similar argument yields

$$S_2 \le 16\pi K\,\alpha^{-4}4^{-j}. \tag{1.9}$$

From Minkowski's inequality

$$S_3 \le \int_0^{2\pi} \sum_{r \ne 0, -1} |\widehat{c}(x + 2\pi r)|^2 \sum_{r \ne 0, -1} \left|\widehat{d}(2^j(x + 2\pi r))\right|^2 dx.$$

If $x \in [0, 2\pi]$ and $r \ge 1$ then from (1.5) we have:

$$\sum_{r \ge 1} |\widehat{c}(x + 2\pi r)|^2 \le 2\pi^{-2}\sum_{r \ge 1} r^{-2} = 1/3,$$

whereas (1.7) implies that

$$\sum_{r \ge 1} \left|\widehat{d}(2^j(x + 2\pi r))\right|^2 \le 2\,4^{-j}\pi^{-2}\sum_{r \ge 1} r^{-2} = 4^{-j}/3.$$

Similarly,

$$\sum_{r \le -2} |\widehat{c}(x + 2\pi r)|^2 \le 2\pi^{-2}\sum_{r \ge 1} r^{-2} = 1/3,$$

and

$$\sum_{r \le -2} \left|\widehat{d}(2^j(x + 2\pi r))\right|^2 \le 2\,4^{-j}\pi^{-2}\sum_{r \ge 1} r^{-2} = 4^{-j}/3,$$

whence we conclude that $S_3 \le (4\pi/9)4^{-j}$. Combining (1.8), (1.9) and the preceding inequality, the assertion follows.

(b) Note that $\widehat{p^{\{6\}}}$ is $\widehat{p^{\{5\}}}$ with $\delta = 1/2$. Since $\widehat{p^{\{5\}}}(x) = 2ix^{-1}e^{-(1/2)xi}(1 - \cos\delta x)$, we see that

$$|\widehat{p^{\{5\}}}(x + 2\pi r)|^2 \le 4|x + 2\pi r|^{-2}, \quad x \ne 2\pi r. \tag{1.10}$$

On the other hand, the inequality $|1 - \cos\delta x| \le (1/2)\delta^2 x^2$ implies that $|\widehat{p^{\{5\}}}(x)| \le \delta^2|x|$; therefore

$$|\widehat{p^{\{5\}}}(x + 2\pi r)|^2 \le \delta^4|x + 2\pi r|^2. \tag{1.11}$$

We now repeat the argument employed in (a), using (1.10) instead of (1.5), (1.11) instead of (1.6), and bearing in mind that $\delta < 1/2$.   □

We now find bounds for the quadratic norms of $q(t)$ and $b(t)$.

Lemma 5   (a) $\|\psi\| \leq 1$; (b) $\|b\| \leq 2(m-1)$.

Proof: (a) [1] Theorem 4.3 implies that the functions N_m are nonnegative. This implies that both g and h are nonnegative. In the proof of [3] Lemma 6(f) we show that

$$\int_{\mathbb{R}} g(t)dt = \int_{\mathbb{R}} h(t)\, dt = (m-1)/2,$$

whence

$$\int_{\mathbb{R}} |q(t)|\, dt \leq m-1.$$

Moreover, $|q(t)| \leq 1$ ([3] Lemma 6(h)). Thus,

$$\int_{\mathbb{R}} |q(t)|^2\, dt \leq \int_{\mathbb{R}} |q(t)|\, dt \leq m-1.$$

Therefore,

$$\int_{\mathbb{R}} |p^{\{i\}}(t)|^2\, dt = (\delta/2(m-1)) \int_{\mathbb{R}} |q(t)|^2\, dt \leq \delta/2, \quad i = 1,2,3,4.$$

This implies that

$$\int_{\mathbb{R}} |\psi(t)|^2\, dt \leq 4\delta/2 + \int_{\mathbb{R}} |p^{\{6\}}(t) - p^{\{5\}}(t)|^2\, dt = 2\delta + (1-2\delta) = 1.$$

(b)

$$\int_{\mathbb{R}} |b(t)|^2\, dt \leq \int_{\mathbb{R}} |b(t)|\, dt = 2\int_{\mathbb{R}} |q(t)|\, dt \leq 2(m-1).   \quad □$$

Theorem 1
(a) *If $j \leq 0$,*

$$\left\| \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\| \leq 2\sqrt{6}(m-1)\|\mathbf{a}\|\, 2^{(j-n)/2}.$$

(b) *Let K be defined as in Lemma 4. If $j > 0$,*

$$\left\| \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\| \leq 8 \left[\sqrt{2}\left(K2^{1-n} + 1/3 \right) + \sqrt{2} + 1/3 \right] \|\mathbf{a}\|_1\, 2^{-j/2}.$$

Proof: Assume first that $j \leq 0$. Applying Lemma 2, Lemma 3, and Lemma 5, we have:

$$\left\| \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\|^2 \leq 2 \sum_{k \in \mathbf{Z}} \|\langle f, \psi_{j,k} \rangle \psi_{j,k}\|^2 = 2\|\psi\|^2 \sum_{k \in \mathbf{Z}} |\langle f, \psi_{j,k} \rangle|^2$$

$$\leq 2 \sum_{k \in \mathbf{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq 6\|f\|^2\, 2^j \leq 6\|b\|^2\|\mathbf{a}\|^2 2^{j-n} \leq 24(m-1)^2\|\mathbf{a}\|^2 2^{j-n}.$$

Assume now that $j > 0$. Setting $b_r^{\{i\}}(t) := a_r b^{\{i\}}(2^n t - 4(m-1)r)$, we see that $f_r(t) = b_r^{\{1\}}(t) + b_r^{\{2\}}(t)$. Thus,

$$\left\| \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\| \leq \sum_{i=1}^{2} \sum_{\ell=1}^{6} \sum_{r=0}^{N-1} \left\| \sum_{k \in \mathbf{Z}} \langle b_r^{\{i\}}, p_{j,k}^{\{\ell\}} \rangle \psi_{j,k} \right\|.$$

Applying Lemma 2 and Lemma 5 as above, we see that

$$\left\| \sum_{k \in \mathbf{Z}} \langle b_r^{\{i\}}, p_{j,k}^{\{\ell\}} \rangle \psi_{j,k} \right\|^2 \leq 2 \sum_{k \in \mathbf{Z}} \left| \langle b_r^{\{i\}}, p_{j,k}^{\{\ell\}} \rangle \right|^2.$$

Since the Fourier transforms of $q(t)$ and $\chi_{[0,2(m-1))}q(t)$ are identical. and the functions $b_r^{\{i\}}$ are of the form $a_r \, q(\alpha t + \beta)$ or $a_r \, \chi_{[0,2(m-1))}(\alpha t + \beta)q(\alpha t + \beta)$ with $|\alpha| = 2^n$, from Lemma 4 we have:

$$\sum_{k \in \mathbf{Z}} \left| \langle b_r^{\{i\}}, p_{j,k}^{\{\ell\}} \rangle \right|^2 \leq 2|a_r|^2 \left(2\sqrt{K}2^{-n} + 1/3 \right)^2 2^{-j}, \quad \ell = 1, 2, 3, 4,$$

and

$$\sum_{k \in \mathbf{Z}} \left| \langle b_r^{\{i\}} . p_{j,k}^{\{\ell\}} \rangle \right|^2 \leq |a_r|^2 2 \left(\sqrt{2} + 1/3 \right)^2 2^{-j}, \quad \ell = 5, 6,$$

whence the assertion readily follows.   □

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